10. $\dfrac{1}{1^{2n}}+\dfrac{1+\frac{1}{2}}{2^{2n}}+\dfrac{1+\frac{1}{2}+\frac{1}{3}}{3^{2n}}+\cdots\cdots$ ナル

無限級数ヲ $\dfrac{1}{1^n}+\dfrac{1}{2^n}+\dfrac{1}{3^n}+\cdots\cdots$ ノイクツカニヨリ表ハス関係式

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$\int \log(1-t)\, t^{s-1}\, dt \quad (s>0)$ ヲ次ノ如キ積分路デ積分シタトスル。先ヅ単位円ヲ1ヨリ出発シテ時計ト逆ノ向キニ一周シ次ニ1ヨリ原点ノ傍近，ソシテ微小円ニテ原点ヲ負ノ向キニ一周シ再ビ1ニ戻ル積分路デアル。(但シ1モ微小円ニヨリ避ケル) ノトキ微小円ヨリノ影響ハ零デアル。又0カラ1マデノ積分ハ往復ノ

ノ差ガ

$$(1-e^{2\pi s i})\int_0^1 \log(1-t)\,t^{s-1}\,dt$$

又単位円Cニ於ケル積分ハ

$$\int_0^{2\pi}\log(1-\cos\theta-i\sin\theta)\,e^{i(s-1)\theta}e^{i\theta}i\,d\theta$$

$$= i\int_0^{2\pi}\log 2\sin\frac{\theta}{2}+i\,\frac{\theta-\pi}{2})\,e^{is\theta}\,d\theta$$

両者ノ和ハCauchyノ定理ニヨリ零ニ等シイカラ

$$i\int_0^{2\pi}\left(\log 2\sin\frac{\theta}{2}+i\,\frac{\theta-\pi}{2}\right)\frac{e^{is\theta}}{e^{2\pi s i}-1}\,d\theta \text{ ハ}$$

$$=\int_0^1 \log(1-t)\,t^{s-1}\,dt$$

$$=-\sum_{n=0}^{\infty}(-1)^n C_{n+2}\,s^n \quad\cdots\cdots\cdots\cdots\cdots (1)$$

茲ニ $C_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\cdots$

然ルニ $\frac{e^{i\theta s}}{e^{2\pi s i}-1}=\sum_{n=0}^{\infty}\frac{B_n(\frac{\theta}{2\pi})}{n!}(2\pi s i)^n$

故ニ(1)ノ両辺ヲsノ冪級数ニ展開シタトキノs^{n-1}ノ係数ヲ比較スレバ

$$(-1)^n C_{n+1}=i\int_0^{2\pi}\left(\log 2\sin\frac{\theta}{2}+i\,\frac{\theta-\pi}{2}\right)\frac{B_n(\frac{\theta}{2\pi})}{n!}(2\pi i)^{n-1}\,d\theta$$

右辺ハ実数デアリ, nガ偶数デアレバ

$$C_{2n+1}=\frac{(2\pi)^{2n-1}}{(2n)!}(-1)^n\int_0^{2\pi}B_{2n}\left(\frac{\theta}{2\pi}\right)\log 2\sin\frac{\theta}{2}\,d\theta$$

又nガ奇数デアレバ

$$C_{2n}=\int_0^{2\pi}\frac{\theta-\pi}{2}\,\frac{B_{2n-1}(\frac{\theta}{2\pi})}{(2n-1)!}(2\pi)^{2n-2}(-1)^{n-1}\,d\theta$$

$$=(-1)^{n-1}\frac{(2\pi)^{2n-2}}{2}\,\frac{1}{(2n-1)!}\int_0^{2\pi}(\theta-\pi)\,B_{2n-1}\left(\frac{\theta}{2\pi}\right)d\theta$$

次ニ $\int\{\log(1-t)\}^2\,t^{s-1}\,dt$ ニ就イテ同様ノコトヲ施セバ

$$i\int_0^{2\pi}\left\{\left(\log 2\sin\frac{\theta}{2}\right)^2-\frac{(\theta-\pi)^2}{4}+i(\theta-\pi)\log 2\sin\frac{\theta}{2}\right\}\frac{e^{i\theta s}}{e^{2\pi s i}-1}\,d\theta$$

$$=\int_0^1\{\log(1-t)\}^2\,t^{s-1}\,dt=2\sum_{n=0}^{\infty}(-1)^n C'_{n+2}\,s^n$$

$$\sum i\int_0^{2\pi}\left\{\left(\log 2\sin\frac{\theta}{2}\right)^2-\frac{(\theta-\pi)^2}{4}+i(\theta-\pi)\log 2\sin\frac{\theta}{2}\right\}\frac{B_n(\frac{\theta}{2\pi})}{n!}(2\pi s i)^{n-1}\,d\theta$$

$$= 2\sum_{n=0}^{\infty}(-1)^n C'_{n+2}\, s^n,$$

茲ニ $C'_n = \frac{1}{2^n} + \frac{1+\frac{1}{2}}{3^n} + \frac{1+\frac{1}{2}+\frac{1}{3}}{4^n} + \cdots$

両辺ノ s^{2n} ノ係数ヲ等シト置イテ

$$2C_{2n+2} = \int_0^{2\pi} \frac{B_{2n+1}\left(\frac{\theta}{2\pi}\right)}{(2n+1)!}(2\pi)^{2n}(-1)^{n-1}(\theta-\pi)\log 2\sin\frac{\theta}{2}\,d\theta$$

$$C_{2n+2} = (-1)^{n-1}\frac{(2\pi)^{2n}}{2}\frac{1}{(2n+1)!}\int_0^{2\pi} B_{2n+1}\left(\frac{\theta}{2\pi}\right)(\theta-\pi)\log 2\sin\frac{\theta}{2}\,d\theta$$

n ノ代リニ $n-1$ ト書キ変形スレバ

$$C'_{2n} = (-1)^n \frac{(2\pi)^{2n-1}}{2}\frac{1}{(2n-1)!}\int_0^{2\pi} B_1\left(\frac{\theta}{2\pi}\right)B_{2n-1}\left(\frac{\theta}{2\pi}\right) l$$

然ルニ $B_1(x)\,B_n(x)$

$$= B_{n+1}(x) + \frac{1}{n+1}\binom{n+1}{2}B_1\,B_{n-1}(x) - \frac{1}{n+1}\binom{n+1}{4}B_2\,B_{n-3}(x)$$
$$+ \cdots + (-1)^{k+1}B_k\frac{\binom{n+1}{2k}}{n+1}B_{n-2k+1}(x) + \cdots$$

(コノ展開ハアラユル多項式ガソノ次数以下ノ次数ノ Bernoulli 多項式ノ一次結合トシテ一意的ニアラハサレル事及ビ $B_n(1-x) = (-1)^n B_n(x)$ カラ $B_n(x)$, $B_{n-2}(x)$, ……ノ項ハ上式デハ不要ナル事更ニ $B_n(1+x) = B_n(x) + nx^{n-1}$ カラ未定係数ヲ決定スル事ニヨリ得ラレル)

故ニ
$$C'_{2n} = (-1)^n\frac{(2\pi)^{2n-1}}{2}\frac{1}{(2n-1)!}\left(\int_0^{2\pi}B_{2n}\left(\frac{\theta}{2\pi}\right)\log 2\sin\frac{\theta}{2}\,d\theta\right.$$
$$\left. + \sum\frac{(-1)^{k+1}}{2n}B_k\binom{2n}{2k}\int_0^{2\pi}B_{2n-2k}\left(\frac{\theta}{2\pi}\right)\log 2\sin\frac{\theta}{2}\,d\theta\right)$$

然ルニ $\int_0^{2\pi}B_{2n-2k}\left(\frac{\theta}{2\pi}\right)\log 2\sin\frac{\theta}{2}\,d\theta = \frac{(-1)^{n-k}(2n-2k)!}{(2\pi)^{2n-2k-1}}C_{2n-2k+1}$

コレ等ヲ代入シテ

$$C'_{2n} = (-1)^n\frac{(2\pi)^{2n-1}}{2}\frac{1}{(2n-1)!}\left\{\frac{(2n)!(-1)^n}{(2\pi)^{2n-1}}C_{2n+1}\right.$$
$$\left. + \sum_{k=1}^{n-1}\frac{(-1)^{k+1}B_k\binom{2n}{2k}}{2n}\frac{(-1)^{n-1}(2n-2k)!}{(2\pi)^{2n-2k-1}}C_{2n-2k+1}\right\}$$

$$= n\,C_{2n+1} - \sum_{k=1}^{n-1}\frac{(2\pi)^{2k}}{2}\frac{B_k}{(2k)!}C_{2n-2k+1}$$

C'_{2n} ノ代リニ

$$\frac{1}{1^{2n}}+\frac{1+\frac{1}{2}}{2^{2n}}+\frac{1+\frac{1}{2}+\frac{1}{3}}{3^{2n}}+\cdots\cdots\left(=C'_{2n}+C_{2n+1}\right)$$

$$=(n+1)\,C_{2n+1}-\sum_{K=1}^{n-1}\frac{(2\pi)^{2K}}{2}\,\frac{B_K}{2K!}\,C_{2n-2K+1}$$

然ニ $\quad C'_{2K}=\dfrac{(2\pi)^{2K}}{2}\,\dfrac{B_K}{2K!}$

$$\therefore\quad \frac{1}{1^{2n}}+\frac{1+\frac{1}{2}}{2^{2n}}+\frac{1+\frac{1}{2}+\frac{1}{3}}{3^{2n}}+\cdots\cdots$$

$$=(n+1)\,C_{2n+1}-\sum_{K=1}^{n-1}C_{2K}\,C_{2n-2K+1}$$

$$=(n+1)\,C_{2n+1}-\sum_{K=1}^{n-1}C_{2K+1}\,C_{2n-2K}$$

又 $C_{2n}=\dfrac{(2\pi)^{2n}}{2}\,\dfrac{B_n}{2n!}$ ナル公式ハ前出ノ

$$C_{2n}=(-1)^{n-1}\,\frac{(2\pi)^{2n-2}}{2}\,\frac{1}{(2n-1)!}\int_0^{2\pi}(\theta-\pi)\,B_{2n-1}\left(\frac{\theta}{2\pi}\right)d\theta$$

ト同ジ積分路デ行ッタ等式カラ出ル $\int_0^{2\pi}B_{2n}\left(\frac{\theta}{2\pi}\right)d\theta$ トカラ計算スレバ出ル。先ニ得タ公式ノ特別ノ場合トシテ

$$\frac{1}{1^2}+\frac{1+\frac{1}{2}}{2^2}+\frac{1+\frac{1}{2}+\frac{1}{3}}{3^2}+\cdots\cdots=2\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\cdots\right)$$

$\log(1-x)$ ノ代リニ $\log(1+x)\tan^{-1}x$ ヲ入レタ積分ヲ用ヒル事ニヨリ先ニ得タ公式ト同様ノ関係ガ類似ノ級数ニ対シ成立スル事ガワカル。